

# METRIC UNIFORMIZATION OF MORPHISMS OF BERKOVICH CURVES

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**ABSTRACT.** We show that the metric structure of morphisms  $f: Y \rightarrow X$  between quasi-smooth compact Berkovich curves over an algebraically closed field admits a finite combinatorial description. In particular, for a large enough skeleton  $\Gamma = (\Gamma_Y, \Gamma_X)$  of  $f$ , the sets  $N_{f, \geq n}$  of points of  $Y$  of multiplicity at least  $n$  in the fiber are radial around  $\Gamma_Y$  with the radius changing piecewise monomially along  $\Gamma_Y$ . In this case, for any interval  $l = [z, y] \subset Y$  connecting a rigid point  $z$  to the skeleton, the restriction  $f|_l$  gives rise to a *profile* piecewise monomial function  $\varphi_y: [0, 1] \rightarrow [0, 1]$  that depends only on the type 2 point  $y \in \Gamma_Y$ . In particular, the metric structure of  $f$  is determined by  $\Gamma$  and the family of the profile functions  $\{\varphi_y\}$  with  $y \in \Gamma_Y^{(2)}$ . We prove that this family is piecewise monomial in  $y$  and naturally extends to the whole  $Y^{\text{hyp}}$ . In addition, we extend the theory of higher ramification groups to arbitrary real-valued fields and show that  $\varphi_y$  coincides with the Herbrand's function of  $\mathcal{H}(y)/\mathcal{H}(f(y))$ . This gives a curious geometric interpretation of the Herbrand's function, which applies also to non-normal and even inseparable extensions.

## 1. INTRODUCTION

### 1.1. Motivation.

1.1.1. *Metric structure and the multiplicity.* Assume that  $f: Y \rightarrow X$  is a finite morphism between nice Berkovich curves (see 2.1.2) over an algebraically closed ground field  $k$ . Note that  $Y$  and  $X$  possess a natural metric and  $f$  is piecewise monomial on intervals  $I \subset Y$  with respect to this metric by [CTT14, Lemma 3.6.8]. Our aim is to find a "finite combinatorial" description of  $f$  as a piecewise monomial map between metric graphs. Note that the slope of  $f|_I$  at  $y$  is the multiplicity  $n_f(y)$  of  $f$  at  $y$  by [CTT14, Lemma 3.5.8], hence the metric structure of  $f$  is described by the multiplicity function  $n_f: Y \rightarrow \mathbf{N}$ , or just by the loci  $N_{f, \geq d}$  of points  $y \in Y$  of multiplicity at least  $d$ .

1.1.2. *The prequel.* If  $f$  is topologically tame, the situation is very simple since  $N_{f, \geq 2}$  is contained in a finite graph. However, the sets  $N_{f, p^d}$  with  $p = \text{char}(\tilde{k})$  can be very large in the topologically wild case, and their structure was absolutely unclear until very recently. The aim of this paper and its prequel [CTT14] was to find a reasonable combinatorial description of the sets  $N_{f, \geq d}$ . In [CTT14], we studied the simplest invariant that distinguishes wild ramification – the different. In particular, we showed that the different function  $\delta_f: Y \rightarrow [0, 1]$  is piecewise monomial, satisfies a balancing condition at type 2 points and relates the genus of

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$Y$  to that of  $X$ . In addition, we showed that the different is "trivialized" by any skeleton of  $f$ , [CTT14, Theorem 6.1.9], and it completely controls the set  $N_{f,p}$  for morphisms of degree  $p$ , [CTT14, Theorem 7.1.4]. Namely,  $N_{f,p}$  is a radial set with center  $\Gamma_Y$  and of radius  $\delta_f^{1/(p-1)}$ .

1.1.3. *This paper.* The first goal of this paper is to prove the radialization theorem that all sets  $N_{f,\geq d}$  are radial with respect to a large enough skeleton  $\Gamma$  of  $f$ , and their  $\Gamma$ -radii are piecewise  $|k^\times|$ -monomial functions on  $\Gamma$ . This is the result mentioned in [CTT14, 1.4], and it is proved in the first half of the paper in a pretty elementary and self-contained way. In particular, we use only very basic properties of the different from [CTT14].

Once the radialization theorem is proved, the second goal is to describe the radii of  $N_{f,\geq d}$  in terms of classical ramification invariants. The information about the radii is equivalently encoded in the piecewise monomial profile function mentioned in the abstract, and we achieve the second goal by interpreting the profile function as the Herbrand's function. In particular, the radii around  $y \in \Gamma_Y$  are directly related to the break points of the higher ramification filtration of  $\mathcal{H}(y)/\mathcal{H}(f(y))$ , see Theorem 4.5.4. Note that to make this rigorous, we have also to extend the classical higher ramification theory to real-valued fields with non-discrete valuations.

## 1.2. Method and main results.

1.2.1. *The splitting method.* All main results are proved by the same splitting method that reduces the general case to the tame and degree- $p$  cases. For example, we independently introduce and study two types of piecewise monomial functions before comparing them: the profile functions and the Herbrand's functions. In both cases we show that

- (1) The function is compatible with compositions (of functions or of field extensions, respectively).
- (2) The function is trivial in the tame case.
- (3) If the degree is  $p$  then the function is described by the different as follows: the slopes are 1 and  $p$  and the break is at  $\delta^{1/(p-1)}$ .

Then the families of such functions  $\varphi_y$  or  $\varphi_{L/K}$  are completely described by these three conditions. For example, in the case of extensions take Galois closure  $F/K$  of  $L/K$ . Then  $\varphi_{L/K}$  is determined by  $\varphi_{F/K}$  and  $\varphi_{F/L}$ . The two other extensions are Galois, hence split into compositions of tame extensions and extensions of degree  $p$ , and hence their Hebrand's functions are determined by (1)–(3). A similar argument works for  $\varphi_y$  after a localization on  $X$ , see Theorem 3.4.2.

1.2.2. *Radialization theorems.* We say that a skeleton  $\Gamma$  *radializes*  $f$  if for any interval  $l = [z, y] \subset Y$  connecting a rigid point  $z$  to the skeleton the restriction  $f|_l$ , viewed as a function  $\varphi_y: [0, 1] \rightarrow [0, 1]$ , depends only on  $y$ . The collection  $\{\varphi_y\}$  is then called the *profile* of  $f$ . It is easy to see that  $\Gamma$  radializes  $f$  if and only if all sets  $N_{f,\geq d}$  are  $\Gamma$ -radial, see Theorem 3.2.10.

Our first main result is that any finite morphism between nice  $k$ -analytic curves is radialized by a large enough skeleton, see Theorem 3.3.11 and Lemma 3.2.14(ii). Moreover, we show that if  $f$  is either a normal covering, or topologically tame, or of degree  $p$  then any skeleton of  $f$  is radializing, see Theorems 3.3.9 and 3.3.7 and Lemma 3.3.2. Note that we establish the topologically tame and degree- $p$  cases

first, and the other claims are deduced by local factorization of  $f$  into morphisms of these two types.

In addition, we provide examples in Section 2.5 of non-radializing skeletons when the degree of  $f$  equals to  $2p$  and  $p^2$ .

1.2.3. *The global profile function.* To complete the combinatorial description of  $f$ , one should also show that the  $\Gamma$ -radii of the sets  $N_{f,\geq d}$  depend piecewise monomially on  $y \in \Gamma$ . Equivalently, one should prove that the profile functions vary piecewise monomially. In fact, we solve a slightly more general problem. Since profile functions are compatible with extensions of skeletons, the radialization theorem implies that to any type 2 point  $y$  there is assigned a profile function  $\varphi_y$  which possesses the following geometric interpretation: if  $l = [z, y]$  is a path starting at a rigid point and approaching  $y$  from a general direction (i.e. from any but finitely many directions) then  $\varphi_y = f|_l$ . We prove that this family depends piecewise monomially on  $y$  and extends to the set  $Y^{\text{hyp}}$  of all non-rigid points, see Theorem 3.4.8.

1.2.4. *Herbrand's function.* It is natural to expect that  $\varphi_y$  is determined by the ramification theory of the field extension  $\mathcal{H}(y)/\mathcal{H}(f(y))$ . We prove that, indeed,  $\varphi_y$  is nothing else but the Herbrand's function of  $\mathcal{H}(y)/\mathcal{H}(f(y))$ . Using the splitting method the proof reduces to the tame and degree- $p$  cases, where the comparison is simple. The only obstacle is that the theory of higher ramification was not developed in the non-discrete case, so our main task is to complete this gap. It is known that the meaningful theory of Herbrand's functions and upper indexed ramification groups exists only for certain classes of extensions. In the classical situation, one considers monogeneous extensions. In the non-discrete case, one should replace this with a sort of an "almost" condition. We introduce in Section 4.2 almost monogeneous extensions and develop for them the theory of upper indexed ramification groups. In addition, we prove that if  $x$  is a point of a  $k$ -analytic curve then any finite extension of  $\mathcal{H}(x)$  is almost monogeneous. On the other hand, we do not know what is the most general class of extensions for which the theory works properly, but see Remark 4.2.3(iii) for a possible candidate.

1.2.5. *Structure of the paper.* In Section 2 we study radial morphisms between open discs. This section is very simple and it serves as a preparation to Section 3, where the radialization theorems are proved. In addition, we extensively study the profile function in Section 3.4. In Section 4, we develop the theory of ramification groups and Herbrand's functions  $\varphi_{L/K}$  for general real-valued fields, and prove in Theorem 4.5.2 that  $\varphi_y = \varphi_{\mathcal{H}(y)/\mathcal{H}(f(y))}$ . In particular, this provides a complete description of the sets  $N_{f,\geq d}$  in terms of the Herbrand's function, see Theorem 4.5.4. Finally, we explain in the end of Section 4.5 how the limit behaviour of  $\varphi_f$  at type 1 and type 2 points can be naturally described in terms of the *logarithmic* Herbrand's function of the corresponding extension of valued fields of height two. Since the latter notion is not developed in this paper (and is missing in the literature in the non-discrete case), we only indicate a justification of this description.

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## 2. RADIAL MORPHISMS BETWEEN OPEN DISCS

### 2.1. Conventions.

2.1.1. *Ground field.* Throughout the paper  $k$  is an algebraically closed complete real-valued field. The valuation can be trivial, though in this case our results are trivial too. By  $p \in \{1, 2, 3, 5, 7, \dots\}$  we denote the exponential characteristic of  $\tilde{k}$ .

2.1.2. *Nice compact curves.* For shortness, a *nice compact curve* means a compact connected separated quasi-smooth strictly  $k$ -analytic curve throughout this paper. For technical convenience, we include the connectedness assumption; but it can be removed in our main results just by working separately with the connected components.

2.1.3. *Multiplicity.* Assume that  $f: Y \rightarrow X$  is a finite morphism of nice compact curves. Given a point  $y \in Y$  with  $x = f(y)$  we have that  $m_y = m_x^e \mathcal{O}_{Y,y}$  and we define the *multiplicity* of  $f$  at  $y$  to be equal to  $n_y = e \cdot [\mathcal{H}(y) : \mathcal{H}(x)]$ . The function  $n_f: Y \rightarrow \mathbf{N}$  sending  $y$  to  $n_y$  will be called the *multiplicity function* associated with  $f$ . By  $N_{f,d}$  or simply  $N_d$  we will denote the *multiplicity- $d$  locus*, i.e. the set of points  $y \in Y$  with  $n_f(y) = d$ .

2.1.4. *Open discs.* By an *open disc* we will always mean an open disc of an integral radius, i.e. an analytic space isomorphic to the open unit disc in  $\mathbf{A}_k^1$ .

2.2. **The PL structure.** First, we recall some well known facts related to the piecewise linear structure of open discs. In fact, this will be a piecewise monomial structure since we use the multiplicative notation.

2.2.1. *Piecewise  $k^\times$ -monomial functions.* As in [Ber04, Section 1] or [CTT14, 3.6.3], by a piecewise  $|k^\times|$ -monomial function on an interval  $I \subseteq \mathbf{R}_+$  we mean a function  $h: I \rightarrow \mathbf{R}_+$  such that  $I$  is a finite union of intervals  $I_i$  and  $h|_{I_i}$  is a monomial function of the form  $c_i t^{n_i}$  with  $c_i \in |k^\times|$  and  $n_i \in \mathbf{Z}$ ; in particular,  $h$  is continuous. The integers  $n_i$  will be called *degrees* or *slopes* of  $h$ .

2.2.2. *Radius function.* Assume that  $D$  is an open disc. By a *monic coordinate* on  $D$  we mean any element  $t \in \Gamma(\mathcal{O}_D)$  that induces an isomorphism of  $D$  with the unit open disc. The radius function  $r_t(x) = \inf_{c \in k} |t - c|_x$  associated with  $t$  is independent of the choice of  $t$ , and we call it the (monic) radius function of  $D$  and denote  $r_D$ .

2.2.3. *Intervals.* For any point  $x \in D$ , by  $l_x$  we denote the upward interval starting at  $x$ , i.e.  $l_x$  is semiopen,  $x$  is the endpoint of  $l_x$  and  $l_x$  is relatively compact in  $D$ . The radius function  $r_D$  induces a homeomorphism of  $l_x$  onto the interval  $[r_D(x), 1)$ , that we call the *radius parametrization* of  $l_x$ . In particular, if  $x$  is a rigid point then  $l_x$  is identified with the interval  $[0, 1)$  and given any function  $h: D \rightarrow \mathbf{R}$  we will denote by  $h_x: [0, 1) \rightarrow \mathbf{R}$  the restriction of  $h$  onto  $l_x$ .

**2.2.4. Restriction of morphisms onto intervals.** If  $f: E \rightarrow D$  is a morphism of open discs,  $y \in E$  and  $x = f(y)$  then  $f$  maps  $l_y$  to  $l_x$ . In particular, if  $y$  is rigid then  $x$  is rigid too and using the radius parameterizations we obtain a map  $[0, 1) = l_y \rightarrow l_x = [0, 1)$  that will be denoted  $f_y$ .

**Lemma 2.2.5.** *Assume that  $f: E \rightarrow D$  is a finite morphism of discs and  $y \in E$  is a rigid point. Then,*

- (i)  $f_y$  is a piecewise  $k^\times$ -monomial function that bijectively maps  $[0, 1)$  onto itself.
- (ii) The right logarithmic derivative of  $f_y$  coincides with the restriction of the multiplicity function  $n_f$  onto  $l_y$ .

*Proof.* Choose monic parameters of  $E$  and  $D$  so that  $y$  and  $f(y)$  become the origins. Then  $f$  is given by a series  $\phi(t) = \sum_{i=1}^{\infty} c_i t^i$ . Let  $z$  be the point of  $l_y$  of radius  $r$ . Then  $n_f(z)$  is the maximal number  $m$  such that  $\max_i |c_i| r^i = |c_m| r^m$ . In particular, it follows that  $n_f$  is an increasing step function.

Let  $z'$  be the point of  $l_y$  of radius  $r'$  such that  $r < r' < 1$ . Taking  $r'$  close enough to  $r$  we can achieve that  $a_m t^m$  is the dominant term of  $\phi(t)$  on the interval  $[z, z']$ , and then  $f_y = |c_m t^m|$  on  $[z, z']$ . This shows that  $f_y$  is a strictly increasing piecewise  $k^\times$ -monomial function and its the right logarithmic derivative equals to  $m = n_f$  on  $[z, z']$ . The bijectivity of  $f_y$  follows from the fact that  $\lim_{r \rightarrow 1} f_y(r) = 1$  as  $f$  is surjective.  $\square$

### 2.3. Radial morphisms.

**2.3.1. Radial functions.** We say that a function  $h: D \rightarrow \mathbf{R}$  is *radial* if it factors through the radius function, say,  $h(x) = \varphi(r(x))$  for a real-valued function  $\varphi$  on  $[0, 1)$ . We call  $\varphi$  the *profile* of  $h$ ; it will be denoted  $\varphi_h$  when needed.

**2.3.2. Radial morphisms.** A morphism  $f: Y \rightarrow X$  between open discs is called *radial* if the real-valued function  $r_X \circ f$  on  $Y$  is radial. This happens if and only if there exists a function  $\varphi: [0, 1) \rightarrow [0, 1)$  such that  $r_X \circ f = \varphi \circ r_Y$ . We call  $\varphi = \varphi_f$  the *profile* of  $f$ .

**2.3.3. A criterion for being radial.** A geometric meaning of being radial is described in the following lemma, where the notation  $h_y$  and  $f_y$  are as in 2.2.3–2.2.4.

**Lemma 2.3.4.** (i) *Assume that  $D$  is an open disc and  $h: D \rightarrow \mathbf{R}$  is a function such that for any type 4 point  $x$  the restriction of  $h$  onto  $l_x$  is continuous at  $x$ . Then  $h$  is radial if and only if the functions  $h_y$  coincide for all rigid points  $y$ . In this case,  $h_y$  is the profile function of  $h$ .*

(ii) *Assume that  $f: Y \rightarrow X$  is a morphism of open discs. Then  $f$  is radial if and only if the maps  $f_y$  coincide for all rigid points  $y$ . In this case,  $f_y$  is the profile function of  $f$ .*

*Proof.* Let  $D'$  be obtained by removing from  $D$  all points of type 4. By the continuity assumption,  $h$  is radial if and only if its restriction onto  $D'$  is radial. Since  $D'$  is covered by the intervals  $l_y$  with  $y$  a rigid point, the claim of (i) becomes obvious. The second claim is proved similarly, but this time no continuity assumption is needed because  $f$  is automatically continuous.  $\square$

**2.3.5. Radial morphisms and the multiplicity function.** It turns out that to check that a morphism is radial it suffices to check that a single integer-valued function, the multiplicity function, is radial.

**Lemma 2.3.6.** *A morphism between open discs  $f: Y \rightarrow X$  is radial if and only if the multiplicity function  $n_f$  is radial. In this case, the profile of  $n_f$  is the logarithmic derivative from the right of the profile of  $f$ .*

*Proof.* Set  $h = n_f$  for shortness. Note that the criterion of Lemma 2.3.4(i) applies to  $h$  because its restriction onto any interval  $l_y$  can be discontinuous only at type 2 points by [CT14, Lemma 3.6.10]. Therefore the lemma follows from Lemma 2.3.4 and the fact that for any rigid point  $y \in Y$  the logarithmic derivative of  $f_y$  from the right coincides with  $h_y$  by Lemma 2.2.5.  $\square$

**2.3.7. Composition.** Radial morphisms satisfy the two out of three property with respect to compositions.

**Lemma 2.3.8.** *Let  $f: Z \rightarrow Y$  and  $g: Y \rightarrow X$  be finite morphisms between open discs and  $h = g \circ f$ . If any two morphisms from the triple  $f, g, h$  are radial then all three are so. In this case, the profiles are related by the rule  $\varphi_h = \varphi_g \circ \varphi_f$ .*

*Proof.* If  $z \in Z$  is a rigid point,  $y = f(z)$  and  $x = g(y)$  then  $g_y \circ f_z = h_z$ . Since the functions  $f_z, g_y, h_z$  are invertible by Lemma 2.2.5(i), two of them determine the third one. The assertion now follows from Lemma 2.3.4(ii).  $\square$

**2.3.9. Restriction onto smaller discs.** If  $X$  is an open disc and  $X' \subseteq X$  is an open subdisc of radius  $c$  then  $r_{X'} = c^{-1}r_X|_{X'}$ . This obvious observation implies that the property of being radial is preserved under restrictions onto smaller discs:

**Lemma 2.3.10.** *Let  $X$  be an open disc with an open subdisc  $X'$  of radius  $c$ .*

*(i) If  $h: X \rightarrow \mathbf{R}$  is a radial function of profile  $\varphi(t)$  then  $h|_{X'}$  is a radial function of profile  $\varphi(ct)$ .*

*(ii) Assume that  $Y$  is an open disc and  $f: X \rightarrow Y$  is a radial morphism of profile  $\varphi$ . Then  $Y' = f(X')$  is an open disc and the restriction morphism  $f': X' \rightarrow Y'$  is radial with profile function  $\varphi' = a^{-1}\varphi(ct)$ , where  $a$  is the radius of  $Y'$  in  $Y$ .*

*Proof.* The arguments are simple and similar, so we only check (ii). Since  $r_Y \circ f = \varphi \circ r_X$ , we have that  $ar_{Y'} \circ f' = \varphi \circ cr_{X'}$  and hence  $r_{Y'} \circ f' = \varphi' \circ r_{X'}$ .  $\square$

## 2.4. Criteria of radially.

**2.4.1. Finite morphisms of discs.** Let  $f: Y \rightarrow X$  be a finite map of open discs. Choose monic coordinates  $t$  and  $x$ , then  $f$  is described by sending  $x$  to a series  $\phi(t) = \sum c_i t^i$  with  $\max_i |c_i| = 1$  and  $|c_0| < 1$ . Choosing the coordinates so that  $f$  respects the origins we can also achieve that  $c_0 = 0$ . The degree  $d = \deg(f)$  is the minimal number with  $|c_d| = 1$ .

**Lemma 2.4.2.** *Let  $f: Y \rightarrow X$  be a finite morphism of open discs and  $d = \deg(f)$ .*

*(i) If  $f$  is étale then either  $d = 1$  or  $p > 1$  and  $d \in p\mathbf{N}$ .*

*(ii) If  $f$  is a Galois covering then  $d \in p\mathbf{N}$ .*

*Proof.* If  $f$  is given by  $\phi(t)$  as above then  $f$  is étale if and only if  $\phi'(t) = \sum_{i=1}^{\infty} ic_i t^{i-1}$  is invertible on  $Y$ . This happens if and only if  $|c_1| \geq |ic_i|$ . If  $d > 1$  then  $|c_1| < 1$  and hence  $|c_i| < 1$  for any  $i \notin p\mathbf{N}$ . This proves (i).

To prove (ii) we consider a  $p$ -Sylow subgroup  $H$  of  $\text{Gal}(Y/X)$ . Since  $Y/H$  is a finite image of an open disc, it is an open disc, and by the first claim either  $Y/H = X$  or  $p$  divides the degree of  $Y/H$  over  $X$ . The second case is impossible, hence  $Y/H = X$  and we obtain that  $\text{Gal}(Y/X) = H$  is a  $p$ -group.  $\square$

2.4.3. *Galois coverings.* The degree- $p$  case is easily studied by hands.

**Lemma 2.4.4.** *Let  $f: Y \rightarrow X$  be a finite étale morphism of discs. If  $f$  is of degree  $p$  then it is radial and  $n_f(y) \in \{1, p\}$  for any  $y \in Y$ .*

*Proof.* We can assume that  $f$  is given by a series  $\phi(t) = \sum_{i=1}^{\infty} c_i t^i$ . Note that  $\phi$  satisfies the following condition: (\*)  $|c_1| > |c_i|$  when  $(p, i) = 1$ ,  $|c_1| > |pc_p|$ , and  $|c_p| = 1 > |c_i|$  for any  $i < p$ . This condition implies that on the upward interval  $l_O$  starting at the origin  $O \in Y$ , the dominant term of  $\phi$  is either  $c_1 t$  or  $c_p t^p$  and the radius of the breaking point satisfies  $|c_1| r = |c_p| r^p = r^p$ , and so  $r = |c_1|^{1/(p-1)}$ . Moreover, (\*) is invariant under translations of the disc because, by a direct inspection,  $\phi(t+b)$  satisfies (\*) for any  $b \in k$  with  $|b| < 1$ . Therefore,  $n_f(z) = 1$  if  $r(z) < |c_1|^{1/(p-1)}$  and  $n_f(z) = p$  otherwise. It remains to use Lemma 2.3.6.  $\square$

**Corollary 2.4.5.** *If  $f: Y \rightarrow X$  is a Galois covering of degree  $d$  of an open disc by an open disc, then the morphism  $f$  is radial,  $d = p^m$  and  $n_f(y) | p^m$  for any  $y \in Y$ .*

*Proof.* Note that  $d = p^m$  by Lemma 2.4.2(ii) and hence  $G = \text{Gal}(Y/X)$  is a  $p$ -group. Thus,  $G$  is solvable and we can factor  $f$  into a tower of étale coverings of degree  $p$ . The latter are radial by Lemma 2.4.4, hence  $f$  is radial by Lemma 2.3.8. For any  $f_i$ , the values of the function  $n_{f_i}$  lie inside  $\{1, p\}$ , hence  $n_f$  takes values in the set of products  $n_1 n_2 \dots n_m$  with  $n_i \in \{1, p\}$ .  $\square$

2.5. **Non-radial examples.** After proving affirmative results about radially, let us discuss the limitations. For this we will construct a few non-radial examples. Throughout Section 2.5 we assume, for concreteness, that  $\text{char}(k) = p > 2$ . Nevertheless, it is easy to see that analogous examples also exist when  $p = 2$  or/and the characteristic is mixed.

2.5.1. *A framework.* We will describe polynomials  $\phi(t)$  that define non-radial morphisms  $f: Y \rightarrow X$ . In particular, we will see that  $\deg(f)$  can be a  $p$ th power or of the form  $np$  with  $(n, p) = 1$ . In all cases, we will exhibit a point  $z$  on the upward interval  $l_O$  starting at the origin  $O \in Y$  such that  $n_f(z) \notin p^{\mathbf{N}}$ . On the other hand it follows easily from Theorem 4.5.4(i) below that for any radial morphism  $f$  the values of  $n_f$  lie in  $p^{\mathbf{N}}$ . (There is no circular reasoning here since the examples are given for illustration only and they will not be used in the sequel.)

2.5.2. *Degree  $2p$ .* Take  $\phi = t^{2p} + c_1 t$  with  $|c_1| < 1$ . Then  $n_f$  takes the values 1 and  $2p$  on the interval  $l_O$ . In particular,  $f$  is not radial.

2.5.3. *Degree  $p^2$ .* Take  $\phi = t^{p^2} + c_{2p} t^{2p} + c_1 t$  such that  $|c_{2p}| < 1$  and  $|c_1/c_{2p}|^{1/(2p-1)} < |c_{2p}|^{1/(p^2-2p)}$ . Then  $n_f$  takes the values 1,  $2p$ ,  $p^2$  on  $l_O$ , hence  $f$  is not radial.

2.5.4. *Split points form a radial set.* Finally, choose  $\phi = t^{2p^2} + c_p t^p + c_1 t$  with  $|c_1| < 1$  and

$$r_1 := |c_1/c_p|^{1/(p-1)} < |c_p|^{1/(2p^2-p)}.$$

The above inequality implies that  $n_f$  takes the values  $1, p, 2p^2$  on  $l_O$ , so  $f$  is not radial. On the other hand, for any  $a \in k$  with  $|a| < 1$  we have that

$$\phi(t+a) - \phi(a) = t^{2p^2} + 2a^{p^2}t^{p^2} + c_p t^p + c_1 t.$$

The above inequality on  $r_1$  implies that  $r_1 < |c_p/2a^{p^2}|^{1/(p^2-p)}$  and hence the linear term of  $\phi(t+a) - \phi(a)$  becomes dominant at the radius  $r_1$ . Thus,  $n_f(z) = 1$  if and only if  $r(z) < r_1$ . In other words, the set of non-split points  $N_{f,>1}$  is radial, i.e. consists of all points whose radius exceeds a fixed threshold. Since  $f$  is not radial,  $n_f$  is not radial and hence some set  $N_{f,>d}$  is not radial. In fact,  $n_f$  takes values  $1, p, 2p^2$ , so we necessarily have that  $N_{f,>p}$  is not radial.

### 3. RADIALIZATION THEOREMS

3.1. **Normal coverings.** In this section we fix our terminology about Galois and normal coverings; the material is pretty standard.

3.1.1. *Galois coverings.* Given a finite morphism of nice compact curves  $f: Y \rightarrow X$  we will also say that  $Y$  or  $f$  is a *finite covering* of  $X$ . We say that  $f$  is a *ramified Galois covering* if the cardinality of  $\text{Aut}_X(Y)$  equals to the degree of  $f$ . The word “ramified” means that  $f$  may have ramification but does not have to. *Galois covering* always means étale Galois covering. By *Galois closure* of a finite covering  $Y \rightarrow X$  we mean the minimal ramified Galois covering (if exists)  $Z \rightarrow X$  that factors through  $Y$ .

**Lemma 3.1.2.** *Any finite generically étale covering of nice compact curves  $f: Y \rightarrow X$  possesses a Galois closure  $Z \rightarrow X$ . Moreover,  $Z$  can be realized as the normalization of an irreducible component of  $(Y/X)^d = Y \times_X Y \times_X \cdots \times_X Y$ , the  $d$ -fold fibred product where  $d = \deg(f)$ .*

*Proof.* Removing a finite set of rigid points from  $X$  and removing their preimages from  $Y$  we obtain a finite étale morphism  $f': Y' \rightarrow X'$ . In this case it is standard that the Galois closure of  $f'$  exists and is realized as a connected component  $Z'$  of  $(Y'/X')^d$ . Let  $Z$  be the normalization of the closure of  $Z'$  in  $(Y/X)^d$ . Then  $Z$  is a nice compact curve and  $g: Z \rightarrow X$  is a finite covering. The fact that  $g$  is Galois and minimal follows from the following simple observation: if nice compact curves  $Z$  and  $T$  are finite coverings of  $X$  and  $Z' \subseteq Z$ ,  $T' \subseteq T$  are the preimages of  $X'$  then any  $X$ -morphism  $T' \rightarrow Z'$  extends uniquely to an  $X$ -morphism  $T \rightarrow Z$ .  $\square$

3.1.3. *Radicial coverings.* We say that a finite morphism of nice compact curves  $f: Y \rightarrow X$  is *radicial* if it is a universal homeomorphism. A typical example is the  $n$ th power of the geometric Frobenius morphism  $F^n X \rightarrow X$ , which is glued from the morphisms of the form  $\mathcal{M}(\mathcal{A}) \rightarrow \mathcal{M}(\mathcal{A}^{p^n})$ , where  $p = \text{char}(k) > 0$ . In fact, they exhaust all radicial morphisms between nice compact curves and, moreover, we have the following lemma.

**Lemma 3.1.4.** *Any finite morphism of connected quasi-smooth  $k$ -analytic curves  $Y \rightarrow X$  factors as  $Y = F^n Z \rightarrow Z \rightarrow X$ , where  $Z \rightarrow X$  is a generically étale finite covering.*



*Proof.* The non-smooth locus of  $f: Y \rightarrow X$  is Zariski closed, so either  $f$  is generically étale and there is nothing to prove or  $f$  is nowhere étale. In the second case it suffices to prove that  $f$  factors as  $Y = FT' \rightarrow T' \rightarrow X$ , because then induction on the degree of  $f$  completes the argument. Let us prove the latter claim. Since it is  $G$ -local on  $X$ , we can assume that  $X = \mathcal{M}(\mathcal{A})$  is affinoid. Then  $Y = \mathcal{M}(\mathcal{B})$  is affinoid too and our claim reduces to showing that  $\mathcal{A} \subseteq \mathcal{B}^p$ . Furthermore, set  $K = \text{Frac}(\mathcal{A})$  and  $L = \text{Frac}(\mathcal{B})$ . Since  $\mathcal{B}/\mathcal{A}$  is a finite extension of Dedekind domains it suffices to show that  $K \subseteq L^p$ .

For any non-rigid point  $y \in Y$  with  $x = f(y)$  we have that  $\kappa(y)$  is not étale over  $\kappa(x)$ . Thus the extension  $\kappa(y)/\kappa(x)$  is inseparable, and since  $\kappa(y)$  is a factor of  $L \otimes_K \kappa(x)$  we obtain that  $L/K$  is inseparable. Thus, it suffices to show that the  $p$ -rank of  $L$  is 1, i.e.  $[L : L^p] = p$ . By noether normalization,  $Y$  is finite over a disc, hence  $\mathcal{B}$  is finite over  $\mathcal{C} = k\{t\}$ . Obviously,  $\mathcal{C}$  is of rank  $p$  over  $\mathcal{C}^p = k\{t^p\}$ , and hence  $\text{Frac}(\mathcal{C})$  is of  $p$ -rank 1. It remains to use that  $L/\text{Frac}(\mathcal{C})$  is finite and the  $p$ -rank of a field is preserved by finite extensions.  $\square$

**3.1.5. Normal coverings.** By a *normal covering* of nice compact curves  $f: Y \rightarrow X$  we mean a finite morphism which is a composition of a radicial morphism and a ramified Galois covering. Normal closure is defined analogously to Galois closure. Lemmas 3.1.4 and 3.1.2 imply the following generalization of the latter.

**Lemma 3.1.6.** *Any finite covering of nice compact curves  $f: Y \rightarrow X$  possesses a normal closure  $Z \rightarrow X$ . Moreover,  $Z$  can be realized as the normalization of the reduction of an irreducible component of  $(Y/X)^d$ , where  $d = \deg(f)$ .*

**3.1.7. Skeletons of finite coverings.** We refer to [CTT14, 3.5.1] for the definition of a skeleton of a nice compact curve. Let  $f: Y \rightarrow X$  be a finite morphism of nice compact curves. If  $f$  is generically étale then a skeleton of  $f$  is defined in [CTT14, 3.5.9] as a simultaneous skeleton  $\Gamma = (\Gamma_Y, \Gamma_X)$ , where  $\Gamma_Y = f^{-1}(\Gamma_X)$  such that  $\Gamma_Y$  contains all ramification points. If  $f$  is not generically étale then this definition makes no sense, so we adjust it as follows. Let  $Y = F^n Z \rightarrow Z \rightarrow X$  be the factorization of  $f$  with a generically étale  $g: Z \rightarrow X$ . Then by a skeleton  $(\Gamma_Y, \Gamma_X)$  of  $f$  we mean any compatible pair of skeletons such that the image of  $\Gamma_Y$  is  $Z$  contains all ramification points of  $g$ . It is easy to see that  $(\Gamma_Y, \Gamma_X)$  is a skeleton of  $f$  if and only if  $(g^{-1}(\Gamma_X), \Gamma_X)$  is a skeleton of  $g$ .

## 3.2. Radial morphisms.

**3.2.1. The retraction  $q_\Gamma$ .** Assume that  $X$  is a nice compact curve with a skeleton  $\Gamma$ . Since  $X \setminus \Gamma$  is a disjoint union of open discs, for any point  $x \in X$  there exists a unique interval  $l_x = [x, q_\Gamma(x)]$  such that  $l_x \cap \Gamma = \{q_\Gamma(x)\}$ . (The interval degenerates to a point when  $x \in \Gamma$ .) Note that  $q_\Gamma: X \rightarrow \Gamma$  is the standard retraction of  $X$  onto  $\Gamma$ . If  $x \in \Gamma$  then the set  $q_\Gamma^{-1}(x) \setminus \{x\}$  is empty if  $x$  is of type 3 or 1 and is a disjoint union of open discs if  $x$  is of type 2.

**3.2.2. The radius function  $r_\Gamma$ .** The skeleton  $\Gamma$  defines a natural radius function  $r_\Gamma: X \rightarrow [0, 1]$  as follows. For a point  $x \in X$  set  $r_\Gamma(x) = \exp(-l)$ , where  $l$  is the length of  $l_x$ . In particular,  $r_\Gamma(x) = 0$  if and only if  $x$  is a rigid point, and, more generally,  $r_\Gamma$  measures the inverse exponential distance of points of  $X$  from  $\Gamma$ .

**Remark 3.2.3.** Any connected component  $D$  of  $X \setminus \Gamma$  is an open disc and the restriction of  $r_\Gamma$  onto  $D$  is the usual radius function of  $D$ .

3.2.4. *Radial sets.* Given a map  $h: \Gamma \rightarrow \mathbf{R}$  we call

$$C(\Gamma, h) = \{x \in X \mid r_\Gamma(x) \geq h(q_\Gamma(x))\}$$

the *radial subset* of  $X$  with *center*  $\Gamma$  and *radius*  $h$ . Also, we say that  $C(\Gamma, h)$  is  $\Gamma$ -*radial*.

3.2.5. *Radial functions.* A function  $h: X \rightarrow \mathbf{R}$  is called  $\Gamma$ -*radial* if for any connected component  $D$  of  $X \setminus \Gamma$ , the restriction  $h|_D$  is radial and its profile  $\varphi_D$  depends only on the limit point  $q(D) \in \Gamma$  of  $D$  in the skeleton, say  $\varphi_D = \varphi_{q(D)}$ . Note that the profile function  $\varphi_q: [0, 1] \rightarrow \mathbf{R}$  naturally extends to  $[0, 1]$  by sending 1 to  $h(q)$ , and by a slight abuse of notation we will denote the extension by the same letter. The collection  $\{\varphi_q\}_{q \in \Gamma^{(2)}}$  is called the *profile* of  $h$ , where  $\Gamma^{(2)}$  denotes the set of type 2 points of  $\Gamma$ . Sometimes, it will be convenient to represent the profile as a single function  $\varphi^{(2)}: \Gamma^{(2)} \times [0, 1] \rightarrow \mathbf{R}$ . If needed, we will mention  $h$  and  $\Gamma$  in the notations, e.g.  $\varphi_h^{(2)}$ .

3.2.6. *Radial morphisms.* Assume that  $f: Y \rightarrow X$  is a finite morphism between nice compact curves and  $\Gamma = (\Gamma_Y, \Gamma_X)$  is a skeleton of  $f$ . We say that  $f$  is  $\Gamma$ -radial if for any connected component  $E$  of  $Y \setminus \Gamma_Y$ , the restriction  $E \rightarrow D = f(E)$  is radial and its profile  $\varphi_E$  depends only on the limit point  $q(E) \in \Gamma_Y$  of  $E$  in the skeleton, say  $\varphi_E = \varphi_{q(E)}$ . Each function  $\varphi_q: [0, 1] \rightarrow [0, 1]$  is a monotonic bijection, so it extends to the whole  $[0, 1]$  by continuity. The extension will be denoted by the same letter and the functions  $\varphi_q$  give rise to a single profile function  $\varphi^{(2)}: \Gamma_Y^{(2)} \times [0, 1] \rightarrow \Gamma_X^{(2)} \times [0, 1]$ .

3.2.7. *Radializing skeletons.* If a morphism  $f$  is  $\Gamma$ -radial then we say that the skeleton  $\Gamma$  *radializes*  $f$ . The same terminology will be used for subsets of  $Y$  and real-valued functions on  $Y$ .

**Remark 3.2.8.** (i) If  $h: X \rightarrow \mathbf{R}$  is a  $\Gamma$ -radial function then one can only extend its profile to a map  $\varphi: \Gamma^{(2)} \times [0, 1] \cup \Gamma \times \{1\} \rightarrow \mathbf{R}$  just by setting  $\varphi(q, 1) = h(q)$ . There is no natural way to define a profile  $\varphi_y$  for  $y \in \Gamma$  of type 3. The situation with profiles of a radial morphism  $f: Y \rightarrow X$  is more interesting. We will later prove that  $\varphi_y$  depends on  $y$  in a piecewise monomial way, and hence  $\varphi^{(2)}$  naturally extends to a map  $\varphi: \Gamma_Y^{(2,3)} \times [0, 1] \rightarrow \Gamma_X^{(2,3)} \times [0, 1]$ .

(ii) Recall that radial functions and morphisms on discs were defined in terms of the radius function. In the same fashion, one can define  $\Gamma$ -radial functions and morphisms in terms of the map  $R_\Gamma = (q_\Gamma, r_\Gamma): X \rightarrow \Gamma \times [0, 1]$ . Namely, a function  $h$  is  $\Gamma$ -radial if it factors through  $R_\Gamma$ , and a morphism  $f: Y \rightarrow X$  is  $\Gamma$ -radial if  $R_X \circ f$  is a radial function on  $Y$ .

3.2.9. *Relation to the multiplicity function.* Results of Section 2 easily extend to morphisms between nice compact curves. We start with the result about the multiplicity.

**Theorem 3.2.10.** *Let  $f: Y \rightarrow X$  be a finite morphism of nice compact curves and let  $\Gamma = (\Gamma_Y, \Gamma_X)$  be a skeleton of  $f$ . Then the following conditions are equivalent:*

- (i) *The morphism  $f$  is  $\Gamma$ -radial.*
- (ii) *The multiplicity function  $n_f$  is  $\Gamma_Y$ -radial.*
- (iii) *The sets  $N_{f, \geq d} := \{y \in Y \mid n_f(y) \geq d\}$  are  $\Gamma_Y$ -radial.*

*Proof.* Equivalence of (i) and (ii) follows from Lemma 2.3.6. Equivalence of (ii) and (iii) follows from the claim that  $n_f$  increases on any interval  $I_y$  in  $Y$ . To check the latter it suffices to consider a finite morphism between open discs, and then the claim was already established in the proof of Lemma 2.3.6.  $\square$

3.2.11. *Composition.* As in the case of discs, radial morphisms are preserved under compositions, but this time we should take the skeletons into account.

**Lemma 3.2.12.** *Let  $f: Z \rightarrow Y$  and  $g: Y \rightarrow X$  be radial morphisms between nice compact curves with the composition  $h: Z \rightarrow X$ . Assume that  $\Gamma_f = (\Gamma_Z, \Gamma_Y)$ ,  $\Gamma_g = (\Gamma_Y, \Gamma_X)$  and  $\Gamma_h = (\Gamma_Z, \Gamma_X)$  are compatible skeletons of  $f$ ,  $g$  and  $h$ , respectively. If two of these skeletons are radializing then all three are radializing and  $\varphi_h^{(2)} = \varphi_g^{(2)} \circ \varphi_f^{(2)}$ .*

*Proof.* This follows from Lemma 2.3.8.  $\square$

3.2.13. *Enlarging the skeleton.* Finally, let us show that radial functions and morphisms are preserved by enlarging the skeleton.

**Lemma 3.2.14.** (i) *Assume that  $X$  is a nice compact curve with skeletons  $\Gamma \subseteq \Gamma'$ . If  $h: X \rightarrow \mathbf{R}$  is a  $\Gamma$ -radial function with profile  $\{\varphi_y\}_{y \in \Gamma^{(2)}}$  then  $h$  is  $\Gamma'$ -radial with profile  $\{\varphi_{y'}\}_{y' \in \Gamma'^{(2)}}$ , where  $\varphi_{y'}(t) = \varphi_{q_\Gamma(y)}(r_\Gamma(y)t)$ .*

(ii) *Assume that  $f: Y \rightarrow X$  is a finite morphism between nice compact curves and  $\Delta \subseteq \Delta'$  are skeletons of  $X$  whose preimages  $\Gamma \subseteq \Gamma'$  in  $Y$  are skeletons. If  $f$  is  $(\Gamma, \Delta)$ -radial with profile  $\{\varphi_y\}_{y \in \Gamma^{(2)}}$  then  $f$  is also  $(\Gamma', \Delta')$ -radial with profile  $\{\varphi_{y'}\}_{y' \in \Gamma'^{(2)}}$ , where  $\varphi_{y'}(t) = r_\Delta(x')^{-1} \varphi_y(r_\Gamma(y')t)$  for each  $y' \in \Gamma'$  with  $y = q_\Gamma(y')$  and  $x' = f(y')$ .*

*Proof.* This follows from Lemma 2.3.10.  $\square$

3.3. **Radialization of morphisms.** Our next aim is to prove that any morphism is radial with respect to a large enough skeleton. In addition, we will see that in certain cases any skeleton is automatically radializing.

3.3.1. *Topologically semi-tame coverings.* We say that a finite morphism between nice compact curves  $f: Y \rightarrow X$  is *topologically semi-tame* if for any  $y \in Y$  the extension  $\mathcal{H}(y)/\mathcal{H}(x)$  is tame. (A more restrictive notion of topologically tame morphisms is introduced in [CTT14, 3.2.3] by requiring that  $n_y$  is invertible in  $\tilde{k}$ . See [CTT14, 3.2.3] for the motivation of this restriction.)

**Lemma 3.3.2.** *Assume that  $f: Y \rightarrow X$  is a finite topologically tame morphism of nice compact curves and  $\Gamma = (\Gamma_Y, \Gamma_X)$  is an arbitrary skeleton of  $f$ . Then  $f$  splits outside of  $\Gamma$ . In particular,  $f$  is  $\Gamma$ -radial and the associated profile  $\{\varphi_y\}$  is trivial, i.e.  $\varphi_y(t) = t$ .*

*Proof.* It suffices to prove that if a finite étale morphism of open discs  $f: E \rightarrow D$  is not an isomorphism then it is not topologically semi-tame. By Lemma 2.4.2(i) the degree of  $f$  is divisible by  $p$ , hence  $f$  is given by a series  $\phi(t) = \sum c_i t^i$  such that  $|c_1| < 1 = \max_i |c_i|$  and  $|c_i| \leq |c_1|$  for  $i \notin p\mathbf{N}$ . Choose  $r$  close enough to 1 so that all dominant terms of  $\phi(t)$  are of the form  $c_{pn} t^{pn}$ , and let  $y$  be the maximal point of the disc around the origin of radius  $r$  and  $x = f(y)$ . Then a direct computation shows that either  $\widetilde{\mathcal{H}(y)}/\widetilde{\mathcal{H}(x)}$  is inseparable or  $|\mathcal{H}(y)^\times|/|\mathcal{H}(x)^\times|$  is divisible by  $p$ . In either case  $\mathcal{H}(y)/\mathcal{H}(x)$  is not tame.  $\square$

**3.3.3. The different.** In [CTT14], a systematic theory of the different function of a morphism  $f: Y \rightarrow X$  is developed. In the sequel, we will need a couple of basic properties of the different that we are going to recall. Given a type 2 point  $y \in Y$ , choose  $t \in \mathcal{H}(y)^\circ$  and  $u \in \mathcal{H}(x)^\circ$  such that  $\tilde{t} \notin \widetilde{\mathcal{H}(y)}^p$  and  $\tilde{u} \notin \widetilde{\mathcal{H}(x)}^p$  and set  $\delta_f(y) = \left| \frac{du}{dt} \right|$ . We claim that  $\delta_f(y)$  is independent of the choice of  $t$  and  $u$ . Indeed, it suffices to show that for any  $t' \in \mathcal{H}(y)^\circ$  with  $\tilde{t}' \notin \widetilde{\mathcal{H}(y)}^p$  we have that  $\left| \frac{dt'}{dt} \right| = 1$  and similarly for  $u$ . Since  $\Omega_{\widetilde{\mathcal{H}(y)}/\widetilde{\mathcal{H}(x)}} = \Omega_{\mathcal{H}(y)^\circ/\mathcal{H}(x)^\circ} \otimes_{\mathcal{H}(y)^\circ} \widetilde{\mathcal{H}(y)}$ , the reduction of  $\frac{dt'}{dt}$  equals to  $\frac{d\tilde{t}'}{d\tilde{t}}$ , which is non-zero since both  $\{\tilde{t}\}$  and  $\{\tilde{t}'\}$  are bases of  $\Omega_{\widetilde{\mathcal{H}(y)}/\tilde{k}}$ . Thus,  $\left| \frac{dt'}{dt} \right| = 1$  as claimed. In fact,  $t$  and  $u$  are tame parameters in the sense of [CTT14, 2.1.2] and so  $\delta_f(y)$  is the different of the extension  $\mathcal{H}(y)/\mathcal{H}(x)$  by [CTT14, Corollary 2.4.6(ii)]. We leave it to the reader to check that a similar construction works for a type 3 point  $y$ , but this time one should take  $t$  with  $|t| \notin |\mathcal{H}(y)^\times|^p$  and similarly for  $u$ , and then  $\delta_f(y) = \left| \frac{du}{dt} \right| \cdot |tu^{-1}|$ .

**Lemma 3.3.4.** *Assume that  $f: Y \rightarrow X$  is a generically étale morphism of nice compact curves and  $I \subset Y$  is an interval consisting of type 2 and 3 points. Then  $\delta_f$  restricts to a piecewise  $|k^\times|$ -monomial function on  $I$ .*

*Proof.* This is a particular case of [CTT14, Corollary 4.1.8].  $\square$

**Remark 3.3.5.** The proof given in [CTT14] is rather straightforward: one covers  $I$  by finitely many intervals that possess global tame coordinates. Once such coordinates are available, the claim reduces to piecewise monomiality of  $|h|$ , where  $h = \frac{du}{dt}$ . Moreover, [CTT14, Corollary 4.1.8] also deals with the slightly more technical case of points of type 4.

**3.3.6. Degree- $p$  coverings.** Now, we can extend Lemma 2.4.4 to nice compact curves.

**Theorem 3.3.7.** *If  $f: Y \rightarrow X$  is a finite morphism of nice compact curves of degree  $p$  then any skeleton  $\Gamma = (\Gamma_Y, \Gamma_X)$  is radializing. In addition, for a point  $y \in \Gamma^{(2)}$  the profile function  $\varphi_y$  is as follows: if  $f$  is radicial then  $\varphi_y(r) = r^p$ , and if  $f$  is generically étale then  $\varphi_y$  has degrees  $p$  and 1 with the breaking point at  $r = \delta^{1/(p-1)}$ . In particular,  $n_f(z) \in \{1, p\}$  for any  $z \in Y \setminus \Gamma_Y$ .*

*Proof.* If  $f$  is radicial then all profiles are of the form  $\varphi_y(r) = r^p$  since  $n_f = p$  everywhere on  $Y$ . So, we can assume that  $f$  is generically étale. Then the assertion follows from [CTT14, Theorem 7.1.4], but we prefer to give a self-contained argument for the sake of completeness.

Let  $E$  be a connected component of  $Y \setminus \Gamma_Y$ . Then  $E$  and  $D = f(E)$  are open discs and  $D$  is a connected component of  $X \setminus \Gamma_X$ . Let  $y \in \Gamma_Y$  and  $x = f(y)$  be the limit points of  $E$  and  $D$ , respectively. By Lemma 2.4.4, the restriction  $g: E \rightarrow D$  of  $f$  is radial and its profile  $\varphi_g$  has degrees 1 and  $p$ . It remains to check that  $\varphi_g$  is as asserted by the theorem. The map  $g$  is given by a series  $u = \phi(t) = \sum c_i t^i$  and we saw in the proof of Lemma 2.4.4 that  $\frac{d \log \varphi_g}{dr}(s) = n_g(s) = 1$  if  $s < |c_1|^{1/(p-1)}$  and  $n_g(s) = p$  otherwise. It remains to observe that for any point  $z \in l_O$ , where  $l_O$  is the upward interval in  $E$  starting at the origin  $O \in E$ ,

$$\delta_f(z) |ut^{-1}|_z = \left| \frac{du}{dt} \right|_z = |c_1|.$$

So, by Lemma 3.3.4 this equality also holds at  $y$ , i.e.  $\delta_f(y) = |c_1 tu^{-1}|_y = |c_1|$ .  $\square$

3.3.8. *Normal coverings.* Normal coverings of nice compact curves can be studied by splitting.

**Theorem 3.3.9.** *Assume that  $f: Y \rightarrow X$  is a normal covering of nice compact curves. Then any skeleton  $\Gamma = (\Gamma_Y, \Gamma_X)$  is radializing and, in addition,  $n_f(y) \in p^{\mathbf{N}}$  for any  $y \in Y \setminus \Gamma_Y$ .*

*Proof.* By Lemma 3.2.12, it suffices to prove this assertion in two cases:  $f$  is radicial, and  $f$  is a ramified Galois covering. The first case is obvious, since  $n_f$  is constant and equals to a power of  $p$ . So, we can assume that  $f$  is a ramified Galois covering. For any connected component  $D$  of  $Y \setminus \Gamma_Y$  the induced morphism  $f_D: D \rightarrow f(D)$  is a Galois covering whose Galois group  $G_D$  can be identified with a subgroup of  $G = \text{Gal}(Y/X)$ , the decomposition group of  $D$ . In particular, Corollary 2.4.5 implies that  $f_D$  is radial,  $|G_D| = p^m$  and  $n_f(y) \in p^{\mathbf{N}}$  for any  $y \in D$ .

To show that  $\Gamma$  is radializing, it suffices to prove that if  $D$  and  $E$  are connected components of  $Y \setminus \Gamma_Y$  with the same limit point in  $\Gamma_Y$  then the radial morphisms  $f_D$  and  $f_E$  have the same profile function. Since  $G_D$  is a  $p$ -group, it can be embedded into a  $p$ -Sylow subgroup  $H$  of  $G$ . Factor  $f$  as  $Y \xrightarrow{a} Y/H \xrightarrow{b} X$  and note that  $b$  induces an isomorphism  $a(D) \xrightarrow{\sim} f(D)$  since  $H$  contains the decomposition group of  $D$ . Thus, the radial morphisms  $f_D$  and  $a_D: D \rightarrow a(D)$  are isomorphic, and hence have the same profile.

In the same way, there is a  $p$ -Sylow subgroup  $H'$  containing  $G_E$ . Since  $H$  and  $H'$  are conjugate, there is a component  $E'$  of  $Y \setminus \Gamma_Y$  conjugated to  $E$  and such that  $H$  contains the decomposition group of  $E'$ . As above,  $f_{E'}: E' \rightarrow f(E')$  and  $a_{E'}: E' \rightarrow a(E')$  are isomorphic and hence have the same profile. But the morphism  $a$  is radial by Theorem 3.3.7 and the solvability of  $H$ . Therefore  $a_{E'}$  and  $a_D$  have the same profile and we obtain that  $f_D$  and  $f_{E'}$  have the same profile. It remains to note that  $f_{E'}$  and  $f_E$  are isomorphic via a conjugation, hence their profiles coincide too.  $\square$

3.3.10. *General finite coverings.* The above theorem allows to easily construct a radializing skeleton for any finite covering.

**Theorem 3.3.11.** *Any finite morphism of nice compact curves  $f: Y \rightarrow X$  possesses a radializing skeleton. Moreover, if  $g: Z \rightarrow X$  is the normal closure of  $f$  with the factorization morphism  $h: Z \rightarrow Y$  and  $(\Gamma_Z, \Gamma_X)$  is any skeleton of  $g$ , then  $(h(\Gamma_Z), \Gamma_X)$  is a radializing skeleton of  $f$ .*

*Proof.* Set  $\Gamma_Y = h(\Gamma_Z)$ , then  $\Gamma = (\Gamma_Y, \Gamma_X)$  is a skeleton of  $f$  and  $(\Gamma_Z, \Gamma_Y)$  is a skeleton of  $h$ . Indeed, it suffices to show that  $Y \setminus \Gamma_Y$  is a disjoint union of open discs and this follows from the fact that if  $E \rightarrow D$  is a finite morphism with  $E$  an open disc then  $D$  is an open disc too. Since  $g$  and  $h$  are normal coverings, they are radial with respect to the corresponding skeletons. Therefore,  $f$  is  $\Gamma$ -radial by Lemma 3.2.12.  $\square$

**Remark 3.3.12.** At this stage it is easy to show that if  $\Gamma$  is a radializing skeleton then  $n_f(y) \in p^{\mathbf{N}}$  for any  $y \in Y \setminus \Gamma_Y$ . We postpone this until 4.5.3 for expository reasons.

3.4. **The profile function.** So far, we only used profiles as a tool in proving radialization results. Studying fine properties of profiles is the aim of this sections.

3.4.1. *The profile function on  $Y^{(2)}$ .* Let  $f : Y \rightarrow X$  be a finite morphism of nice compact curves. By Theorem 3.3.11 and Lemma 3.2.14, any sufficiently large skeleton of  $f$  is radializing. Obviously, profiles are compatible with enlarging of skeletons: if  $\Gamma \subseteq \Gamma'$  are radializing then  $\varphi_{\Gamma, y} = \varphi_{\Gamma', y}$  for any  $y \in \Gamma^{(2)}$ . Therefore, we obtain a global profile function  $\varphi^{(2)} : Y^{(2)} \times [0, 1] \rightarrow X^{(2)} \times [0, 1]$ , where  $Y^{(2)}$  denotes the set of all type 2 points of  $Y$ .

**Theorem 3.4.2.** *Keep the above notation and let  $y \in Y$  be a type 2 point,  $L = \mathcal{H}(y)$  and  $K = \mathcal{H}(f(y))$ . The profile  $\varphi_y$  is an invariant of the extension of valued fields  $L/K$ , say  $\varphi_y = \varphi_{L/K}$ , which is determined by the following conditions:*

- (a)  $\varphi$  is transitive, i.e.  $\varphi_{L/K} = \varphi_{F/K} \circ \varphi_{L/F}$  for any intermediate field  $F$ .
- (b)  $\varphi_{L/K}$  is trivial when  $L/K$  is tame.
- (c)  $\varphi_{L/K}(r) = r^p$  if  $L/K$  is inseparable of degree  $p$ .
- (d) If  $L/K$  is separable of degree  $p$  and with different  $\delta$  then  $\varphi_{L/K}(r)$  has degrees 1 and  $p$  and the breaking point is  $r = \delta^{1/(p-1)}$ .

*Proof.* The profile functions satisfy conditions (b), (c) and (d) by Lemma 3.3.2 and Theorem 3.3.7. It follows from [Ber93, Theorem 3.4.1] that for any  $K \subseteq F \subseteq L$  we can shrink  $X$  and  $Y$  so that  $f$  factors through  $g : Y \rightarrow Z$  with  $\mathcal{H}(g(y)) = F$ . Hence condition (a) follows from Lemma 3.2.12.

On the other hand, these conditions determine the invariant because any normal extension splits into composition of a tame extension and extensions of degree  $p$ . In particular, this implies that  $\varphi_y$  is determined by  $L/K$ .  $\square$

**Remark 3.4.3.** As we saw, there is at most one invariant of extensions of valued fields that satisfies the four conditions from Theorem 3.4.2. Existence is not so obvious and requires some restrictions on the fields. In the classical case when the valuations are discrete and the residue fields are perfect, it is well known that the Herbrand's function satisfies these conditions. We will show in Section 4 that the theory of higher ramification groups extends to the fields  $\mathcal{H}(y)$ , where  $y$  is a point on a  $k$ -analytic curve, and then it will automatically follow that  $\varphi_y$  coincides with the Herbrand's function of  $\mathcal{H}(y)/\mathcal{H}(x)$ .

3.4.4. *Extension of  $\varphi^{(2)}$ .* As in [CTT14, 3.3.2], let  $X^{\text{hyp}}$  denote the sets of non-rigid points. Our next aim is to show that  $\varphi^{(2)}$  extends to a piecewise  $|k^\times|$ -monomial profile function on the whole set  $X^{\text{hyp}}$ .

3.4.5. *The normal case.* We start with the following particular case.

**Lemma 3.4.6.** *If  $f : Y \rightarrow X$  is a normal covering of nice compact curves and  $\Gamma = (\Gamma_Y, \Gamma_X)$  is a radializing skeleton, then the profile function  $\varphi_\Gamma^{(2)} : \Gamma_Y^{(2)} \times [0, 1] \rightarrow \Gamma_X^{(2)} \times [0, 1]$  is piecewise  $|k^\times|$ -monomial, that is, it extends by continuity to a piecewise  $|k^\times|$ -monomial map  $\varphi_\Gamma^{(2,3)} : \Gamma_Y^{(2,3)} \times [0, 1] \rightarrow \Gamma_X^{(2,3)} \times [0, 1]$ .*

*Proof.* If  $f$  is radicial then the claim is trivial since each  $\varphi_y(t)$  is of the form  $t^{p^n}$  where  $\deg(f) = p^n$ . In general,  $f$  factors into the composition of a radicial morphism and a ramified Galois covering. Since profiles are compatible with compositions by Lemma 3.2.12, it suffices to consider the case when  $f$  is a ramified Galois covering. Note also that it suffices to establish the following claim:

*Assume that  $e = (u, v)$  is an (open) edge in  $\Gamma_Y$  between vertices  $u$  and  $v$ , and let  $e' = f(e)$  be its image in  $\Gamma_X$ . Then the restriction  $\varphi_e : e^{(2)} \times [0, 1] \rightarrow e'^{(2)} \times [0, 1]$*



of  $\varphi_\Gamma^{(2)}$  is piecewise  $|k^\times|$ -monomial. In addition, if the endpoint  $u$  is of type 2 then the same is true for the interval  $[u, v)$ .

We start with two particular cases. Let  $A$  be the connected component of  $Y \setminus \{u, v\}$  with skeleton  $e$ . Then  $A$  and  $A' = f(A)$  is either an open annulus or a punctured open disc and it follows from [CTT14, Lemma 3.5.8(ii)] that the finite morphism  $A \rightarrow A'$  has constant multiplicity along  $e$ . We denote this number  $n_e$ .

Case 1. Assume that  $(n_e, p) = 1$ . In this case  $\mathcal{H}(y)/\mathcal{H}(x)$  is tame for any  $y \in e$  and  $x = f(y)$ . So the profile is trivial on  $e \cap Y^{(2)}$  by Lemma 3.3.2 and hence it extends to  $e$  trivially. It remains to check that if  $u$  is of type 2 then  $\varphi_u$  is trivial too. We claim that this is indeed the case since  $\mathcal{H}(u)/\mathcal{H}(f(u))$  is tame. It is easy to check the latter claim straightforwardly, but let us use a shortcut: since  $\delta_f = 1$  on  $e$ , we also have that  $\delta_f(u) = 1$  by Lemma 3.3.4, and hence  $\mathcal{H}(u)/\mathcal{H}(f(u))$  is tame by [CTT14, Lemma 4.2.2(ii)].

Case 2. Assume that  $\deg(f) = p$ . In this case, Theorem 3.3.7 tells that each profile function  $\varphi_y$  has degrees 1 and  $p$  and the breaking point equals to  $\delta_f(y)^{1/(p-1)}$ . So, the assertion follows from the fact that  $\delta_f(y)$  is piecewise  $|k^\times|$ -monomial on  $e$  by Lemma 3.3.4.

Now, consider the general case. Let  $G = \text{Gal}(Y/X)$ , let  $H \subseteq G$  be the decomposition group of  $e$ , and let  $S$  be a  $p$ -Sylow subgroup of  $H$ . Then  $f$  splits as  $Y = Y_0 \rightarrow Y_1 \rightarrow \cdots \rightarrow Y_n = Y/S \rightarrow X$  with the first  $n$  morphisms of degree  $p$ . The claim holds for the morphisms  $Y_i \rightarrow Y_{i+1}$  by Case 2 and it holds for the morphism  $Y/S \rightarrow X$  by Case 1. Since profile functions are compatible with compositions by Lemma 3.2.12, the claim holds for  $f$  as well.  $\square$

3.4.7. *Extension to  $X^{\text{hyp}}$ .* Recall that by [CTT14, Lemma 3.6.8], for any interval  $I \subset Y^{\text{hyp}}$  its image  $f(I)$  is a finite graph and the map  $I \rightarrow f(I)$  is piecewise  $|k^\times|$ -monomial.

**Theorem 3.4.8.** *Assume that  $f: Y \rightarrow X$  is a finite morphism of nice compact curves. Then the profile function  $\varphi_f^{(2)}: Y^{(2)} \times [0, 1] \rightarrow X^{(2)} \times [0, 1]$  uniquely extends to a function  $\varphi_f: Y^{\text{hyp}} \times [0, 1] \rightarrow X^{\text{hyp}} \times [0, 1]$  such that for any interval  $I \subset Y^{\text{hyp}}$  the induced map  $I \times [0, 1] \rightarrow f(I) \times [0, 1]$  is piecewise  $|k^\times|$ -monomial.*

*Proof.* Let  $h: Z \rightarrow X$  be the normal closure of  $f$  and let  $g: Z \rightarrow Y$  be the morphism  $h$  factors through. Recall that by Lemma 3.2.12,  $\varphi_h^{(2)} = \varphi_f^{(2)} \circ \varphi_g^{(2)}$ . Therefore, if the assertion holds for  $g$  and  $h$  and  $\varphi_g, \varphi_h$  are the corresponding profiles then the assertion holds for  $f$  and its profile  $\varphi = \varphi_f$  is determined by  $\varphi_h = \varphi_f \circ \varphi_g$ . Therefore, we are reduced to the case when  $f$  is a normal covering.

By Lemma 3.2.12, for any skeleton  $\Gamma$  of  $f$  the profile function  $\varphi^{(2)}$  extends to a piecewise  $|k^\times|$ -monomial function on  $\Gamma_Y$ . These functions are compatible (since they are compatible on the type 2 points), hence it only remains to extend the profile function to type 4 points. So, assume that  $y$  is of type 4. Choose a skeleton  $\Gamma$  of  $f$  and let  $l_y = [y, q]$  be the interval connecting  $y$  with the skeleton. Using the radius parametrization we identify  $l_y$  with  $[r, 1]$ , where  $0 < r = r_\Gamma(y)$ . By Lemma 3.2.14, the profile function on  $(y, q]$  is determined by the profile function of  $q$  as  $\varphi_z(t) = s^{-1}\varphi_q(rt)$ , where  $z \in (y, q]$ ,  $r = r_\Gamma(z)$  and  $s = \varphi_q(r)$ . Clearly, the same formula extends  $\varphi$  to  $z = y$  in a piecewise monomial way.  $\square$

**Remark 3.4.9.** The profile function extends to rigid points as follows. Assume that  $y \in Y$  is a rigid point and let  $I \subset Y$  be an interval starting at  $y$ . Using the

same arguments as above and [CTT14, Theorem 4.6.4] one can show that if  $f$  is not wildly ramified at  $x$  then  $\varphi_z$  is constant for  $z \in I$  close enough to  $y$ , and if  $f$  is not topologically wild at  $y$  then  $\varphi_z$  is trivial for  $z \in I$  close enough to  $y$ . Finally, if  $f$  is wildly ramified at  $y$  then  $\varphi_z$  tends to 0 at  $y$  and its precise asymptotic behaviour can be expressed in terms of valued fields of height two, see Section 4.5.5 below.

#### 4. PROFILE FUNCTION AND HIGHER RAMIFICATION

Unfortunately, many aspects of the theory of valued fields are not developed beyond the discrete case, and it seems that higher ramification is one of them. In this section we try to complete this gap to some extent. We will study the case of real-valued fields with non-discrete valuation. We ignore the discrete-valued case since it is known and requires to distinguish the usual and the logarithmic filtrations and Herbrand's functions, see Remark 4.1.5.

##### 4.1. Higher ramification groups.

4.1.1. *Notation.* In the sequel,  $K$  is a real-valued field, and we assume that the valuation is non-discrete and  $p = \text{char}(\tilde{K}) > 0$ .

4.1.2. *Henselian extensions.* We say that a finite extension of valued fields  $L/K$  is *henselian* if there is a unique extension of the valuation of  $K$  to  $L$ , or, that is equivalent,  $L^\circ/K^\circ$  is integral. Note that, the valued field  $K$  is henselian if and only if any its finite extension is henselian. Only henselian extensions will be considered until Section 4.5, so sometimes we will not mention this assumption.

4.1.3. *The prefix "almost".* We will use the word "almost" in the sense of almost mathematics of [GR03]. An almost property  $P$  for  $K$  means that  $P$  "holds up to something killed" by any element of  $K^{\circ\circ}$ . For example, a homomorphism of  $K^\circ$ -modules  $f: M \rightarrow N$  is an almost isomorphism if both  $\text{Ker}(f)$  and  $\text{Coker}(f)$  are annihilated by  $K^{\circ\circ}$ .

4.1.4. *Ramification groups.* Assume that  $L/K$  is a finite henselian Galois extension of valued fields. Define the *inertia function*  $i_{L/K}: G \rightarrow [0, 1]$  and the associated increasing filtration of  $G = \text{Gal}(L/K)$  as follows

$$i_{L/K}(\sigma) = \sup_{c \in L} |\sigma c - c|, \quad G_r = \{\sigma \in G \mid i_{L/K}(\sigma) \leq r\}.$$

The groups  $G_r$  are called (higher) *ramification groups*.

**Remark 4.1.5.** (i) In the discrete-valued case, one shifts this filtration by  $|\pi_K|$ . In addition, one considers the logarithmic filtration  $G_r^{\log}$  given by the logarithmic inertia function  $i_{L/K}^{\log}(\sigma) = \sup_{c \in L^\circ} |\frac{\sigma c}{c} - 1|$ . The two filtrations are pretty closed; in fact, it is easy to see that  $G_s \subseteq G_s^{\log} \subseteq G_{s|\pi_K|^{-1}}$  for any  $s$ . Moreover, these filtrations coincide when  $\tilde{L}/\tilde{K}$  is separable.

(ii) In the non-discrete case,  $i_{L/K} = i_{L/K}^{\log}$  and there is no need to consider the logarithmic filtration separately.

4.1.6. *Ramification jumps.* We say that  $0 \neq r \in |L^\circ|$  is a *jump* of the ramification filtration if there exists  $\sigma \in G$  such that  $i_{L/K}(\sigma) = r$ . This happens if and only if the group  $G_{<r} = \cup_{s < r} G_s$  is strictly smaller than  $G_r$ .



**4.1.7. Herbrand's function.** If  $L/K$  is as above then the *Herbrand's function*  $\varphi_{L/K}$  is the bijective piecewise monomial function from  $[0, 1]$  to itself whose breaking points  $r_0 > r_1 > \dots > r_n$  are the jumps of the ramification filtration and whose degrees are described as follows: set  $r_{-1} = 1$  (so  $r_{-1} \geq r_0$ ) and  $r_{n+1} = 0$ , then the degree on the interval  $[r_i, r_{i-1}]$  with  $0 \leq i \leq n+1$  equals to  $g_i := |G_{r_i}|$ ; for example,  $g_0 = |G|$  and  $g_{n+1} = 1$ .

**Remark 4.1.8.** (i) Since we work with the multiplicative valuations, it is natural to represent Herbrand's function as a piecewise monomial function. In the additive setting, the Herbrand's function is a piecewise linear function  $\varphi_{L/K}^{\text{add}}$ . The two functions are related as additive and multiplicative valuations:  $\varphi_{L/K}^{\text{add}} = -\log(\varphi_{L/K})$ .

(ii) In the discrete-valued case,  $\varphi_{L/K}$  maps  $[0, |\pi_L|^{-1}]$  to  $[0, |\pi_K|^{-1}]$  and the slope degree on the interval  $[1, |\pi_L|^{-1}]$  equals to  $e_{L/K}$ . Classically, one uses the additive language and normalizes both valuations so that the group of values is  $\mathbf{Z}$  (e.g. in [Ser79, Ch. IV, §3]). In this case, one works with the function  $\frac{1}{e_{L/K}}\varphi_{L/K} : [-1, \infty) \rightarrow [-1, \infty)$  and the slopes are  $g_i/e_{L/K}$ .

**4.1.9. The product formula.** It is easy to compute  $\varphi_{L/K}$  directly: if  $r \in [r_i, r_{i-1}]$  then

$$\varphi_{L/K}(r) = \left(\frac{r}{r_{i-1}}\right)^{g_i} \prod_{0 \leq j < i} \left(\frac{r_j}{r_{j-1}}\right)^{g_j} = r^{g_i} \prod_{0 \leq j < i} r_j^{g_j - g_{j+1}},$$

but the following formula will be more useful.

**Lemma 4.1.10.** *Let  $L/K$  be a finite henselian Galois extension of real-valued fields and  $r \in [0, 1]$ . Then*

$$\varphi_{L/K}(r) = \prod_{\sigma \in G} \max(i_{L/K}(\sigma), r).$$

*Proof.* It suffices to observe that the righthand side is a piecewise monomial function such that  $\varphi_{L/K}(1) = 1$  and the degree on  $[r_i, r_{i-1}]$  equals to  $g_i$ .  $\square$

**4.1.11. The upper indexing.** Using the Herbrand's function one introduces a shifted filtration via  $G_r = G^s$ , where  $s = \varphi_{L/K}(r)$ . Its jumps  $s_i = \varphi_{L/K}(r_i)$  are often called the *upper jumps* as opposed to the lower jumps  $r_i$ .

**Remark 4.1.12.** (i) Although the Herbrand's function and the upper indexing are defined for any Galois extension  $L/K$ , they are really meaningful only when some restrictions on  $L/K$  are imposed, see Section 4.2 below. In this case, the Herbrand's function is transitive in towers and the upper indexed filtration is compatible with passing to the quotients of  $G$ . The latter properties are the main motivation for introducing  $\varphi_{L/K}$  and the shifted filtration.

(ii) As explained in [Ser79, Ch. IV, Remarks 3], the natural group where lower indexes live is  $|L^\times|^\mathbf{Q}$  while the natural group where upper indexes live is  $|K^\times|^\mathbf{Q}$ , so  $\varphi_{L/K}$  can be naturally viewed as a piecewise monomial map between ordered monoids  $|L^\circ|^\mathbf{Q} \rightarrow |K^\circ|^\mathbf{Q}$ . Moreover, this is the only definition making sense for general valued fields, especially of height larger than one. This interpretation also agrees with the fact that the lower indexing is compatible with passing to subgroups of  $G$  while (in good cases) the upper indexing is compatible with quotients. In addition, it illustrates the similarity between  $\varphi_{L/K}$  and the profile functions.

**4.2. Almost monogeneous extensions.** In this section, we introduce a class of extensions  $L/K$  for which the ramification theory can be extended further.

**4.2.1. Monogeneous extensions.** An extension of valued fields  $L/K$  is called *monogeneous* if the extension of integers is so, i.e.  $L^\circ = K^\circ[x]$ . In the classical theory of ramification groups one often assumes that the residue fields are perfect, but one really needs the consequence that the extensions of rings of integers are monogeneous. This is based on the simple observation that for monogeneous extensions, the inertia function can be computed in terms of a generator  $x$  as  $i_{L/K}(\sigma) = |\sigma(x) - x|$ .

**4.2.2. Almost monogeneous extensions.** In the non-discrete case we can extend the above class of extensions as follows: a finite extension  $L/K$  is *almost monogeneous* if for any  $r \in |L^\circ|$  there exists  $x_r \in L^\circ$  and  $a_r \in L^\circ$  such that  $r < |a_r|$  and  $a_r L^\circ \subseteq K^\circ[x_r]$ . An element  $x_r$  will be called an *r-generator*.

**Remark 4.2.3.** (i) We will see that higher ramification theory works fine for the class of almost monogeneous extensions. However, it is not clear if this class satisfies reasonable functoriality properties. For example, I do not know if it is closed under subextensions.

(ii) One may wonder what is the largest class of extensions to which the higher ramification theory extends. Perhaps, these are extensions  $L/K$  such that the module  $\Omega_{L/K}$  is almost cyclic. One can show that this class is closed under passing to subextensions and contains all almost monogeneous extensions and all separable extensions of degree  $p$ .

**4.2.4. Bounds on  $i_{L/K}$ .** Our motivation to introduce  $r$ -generators is that they provide the following control on the inertia function.

**Lemma 4.2.5.** *If  $L/K$  is a finite henselian Galois extension of real-valued fields with an  $r$ -generator  $x$  and  $\sigma \in \text{Gal}(L/K)$  then*

$$|\sigma(x) - x| \leq i_{L/K}(\sigma) \leq r^{-1}|\sigma(x) - x|.$$

*Proof.* Only the right inequality needs a proof. A simple classical computation shows that any  $y \in K^\circ[x]$  satisfies  $|\sigma(y) - y| \leq |\sigma(x) - x|$ . Hence any  $z \in L^\circ$  satisfies  $|\sigma(z) - z| \leq r^{-1}|\sigma(x) - x|$  and we win.  $\square$

**4.2.6. The key lemma.** Now, we are going to establish a key result that relates  $i_{L/K}$  to  $i_{F/K}$  for  $L/F/K$ . In fact, this is the only computation where we directly use that  $L/K$  is almost monogeneous; all other results will use this assumption via the key lemma. The standard proof in the classical case is due to J. Tate; it is short but rather tricky, see [Ser79, Ch. IV, Prop. 3]. Tate's proof extends to almost monogeneous extensions straightforwardly.

**Lemma 4.2.7.** *Assume that  $L/K$  is an almost monogeneous finite henselian Galois extension of real-valued fields and  $F$  is the invariant field of a normal subgroup  $H \subseteq G = \text{Gal}(L/K)$ . Then  $i_{F/K}(\sigma) = \prod_{\tau \rightarrow \sigma} i_{L/K}(\tau)$  for any  $\sigma \in G/H$ .*

*Proof.* Let  $r < 1$  and fix an  $r$ -generator  $u \in L^\circ$  with minimal polynomial over  $F$

$$f(t) = \sum_{j=0}^d a_j t^j = \prod_{\tau \in H} (t - \tau(u)).$$

Consider the  $\sigma$ -translate

$$f^\sigma(t) = \sum_{j=0}^d \sigma(a_j) t^j = \prod_{\tau \rightarrow \sigma} (t - \tau(u))$$

of  $f(t)$ . Substituting  $t = u$  and using Lemma 4.2.5 we obtain that

$$(1) \quad |f^\sigma(u)| \leq \prod_{\tau \rightarrow \sigma} i_{L/K}(\tau) \leq r^{-d} |f^\sigma(u)|.$$

On the other hand, since  $f(t) \in F^\circ[t]$  we obtain that

$$|f^\sigma(u)| = |f^\sigma(u) - f(u)| \leq \max_j |\sigma(a_j) - a_j| \cdot |u|^j \leq i_{F/K}(\sigma).$$

Combining this with the right side of (1) and using that  $r$  can be arbitrarily close to 1, we obtain that  $i_{F/K}(\sigma) \geq \prod_{\tau \rightarrow \sigma} i_{L/K}(\tau)$ .

Let us prove the opposite inequality. Since  $K$  is not discrete-valued it suffices to show that for any  $v \in F^{\circ\circ}$  the inequality  $|v - \sigma(v)| \leq \prod_{\tau \rightarrow \sigma} i_{L/K}(\tau)$  holds. Enlarging  $r$  if necessary and adjusting  $u$ , we can achieve that  $v \in K^\circ[u]$ , say  $v = h(u)$  for  $h(t) \in K^\circ[t]$ . Then  $u$  annihilates  $h(t) - v \in F^\circ[t]$  and therefore  $h(t) - v = f(t)g(t)$  in  $F^\circ[t]$ . Since  $h^\sigma = h$ , we have that  $h(t) - \sigma(v) = f^\sigma(t)g^\sigma(t)$  and substituting  $t = u$  gives  $v - \sigma(v) = f^\sigma(u)g^\sigma(u)$ . Therefore,  $|v - \sigma(v)| \leq |f^\sigma(u)|$ , and using the left side of (1) we obtain that  $|v - \sigma(v)| \leq \prod_{\tau \rightarrow \sigma} i_{L/K}(\tau)$ , as required.  $\square$

**4.2.8. Almost monogeneous valued fields.** We say that a real-valued field  $K$  is *almost monogeneous* (resp. *monogeneous*) if any finite henselian Galois extension  $L/K$  is so. Some examples are listed below. The main conclusion is that if  $y$  is a point on a  $k$ -analytic curve (and  $k$  is algebraically closed) then  $\mathcal{H}(y)$  is almost monogeneous.

**Example 4.2.9.** (i) If  $f_{L/K} = [L : K]$  and  $\tilde{L}/\tilde{K}$  is generated by a single element then  $L/K$  is monogeneous and an element  $x \in L^\circ$  is a generator of  $L^\circ$  if and only if  $\tilde{x}$  generates  $\tilde{L}$  over  $\tilde{K}$ . In particular, if  $K$  is stable,  $\tilde{K}$  has  $p$ -rank 1 and  $|K^\times|$  is divisible then  $K$  is monogeneous. This includes the fields  $\mathcal{H}(y)$ , where  $y$  is a type 2 point on a  $k$ -analytic curve.

(ii) If  $e_{L/K} = [L : K]$  and  $H = |L^\times|/|K^\times|$  is cyclic then  $L/K$  is almost monogeneous and any element  $x \in L^\circ$ , such that  $|x| \geq r$  and  $|x|$  generates  $H$ , is an  $r$ -generator. In particular, if  $K$  is stable and not discrete-valued,  $\tilde{K}$  is algebraically closed and  $|K^\times|^\mathbb{Q}/|K^\times|$  is cyclic then  $K$  is almost monogeneous. This includes the fields  $\mathcal{H}(y)$ , where  $y$  is a type 3 point on a  $k$ -analytic curve.

(iii) If  $K = \mathcal{H}(y)$  and  $y$  is of type 4 then any finite extension  $L/K$  is almost monogeneous. To prove this one should use that by stable reduction,  $L = \mathcal{H}(z)$  with  $z$  a point of a disc. Hence  $L = \widehat{k(t)}$  and by a direct computation one can show that the elements  $t_i = t - a_i$  with  $a_i \in k$  and  $|t - a_i|$  tending to  $\inf_{c \in k} |t - c|$  provide a series of  $r_i$ -generators of  $L/K$  with  $r_i$  tending to 1. Perhaps the easiest way to do this is to extend the ground field from  $k$  to  $K$ : since the completion of  $\text{Frac}(K \otimes_k K)$  is of type 2 or 3 over  $K$ , the claim reduces to one of the cases described in (i) and (ii).

### 4.3. Herbrand's theorem.

4.3.1. *Index transition.* To compare  $i_{L/K}$  and  $i_{F/K}$ , for any  $\sigma \in G/H$  set

$$j_{L/F/K}(\sigma) = \min_{\tau \rightarrow \sigma} (i_{L/K}(\tau)).$$

**Lemma 4.3.2.** *Let  $L/K$  be as in Lemma 4.2.7 and let  $F$  be the invariant field of a normal subgroup  $H \subseteq G$ . Then  $i_{F/K}(\sigma) = \varphi_{L/F}(j_{L/F/K}(\sigma))$ .*

*Proof.* Choose  $\tau$  above  $\sigma$  such that  $j_{L/F/K}(\sigma) = i_{L/K}(\tau)$ . Then for any  $\lambda \in H$ ,

$$i_{L/K}(\tau\lambda) = \max(i_{L/K}(\lambda), j_{L/F/K}(\sigma)).$$

So, the key lemma 4.2.7 gives that

$$i_{F/K}(\sigma) = \prod_{\tau \rightarrow \sigma} i_{L/K}(\tau) = \prod_{\lambda \in H} \max(i_{L/K}(\lambda), j_{L/F/K}(\sigma)),$$

and by Lemma 4.1.10 the righthand side equals to  $\varphi_{L/F}(j_{L/F/K}(\sigma))$ .  $\square$

**Corollary 4.3.3.** *Let  $L/F/K$  be as above. If  $s = \varphi_{L/F}(r)$  then  $G_r H/H = (G/H)_s$ .*

*Proof.* Note that  $\sigma \in G_r H/H$  if and only if  $j_{L/F/K}(\sigma) \leq r$ . By Lemma 4.3.2, the latter happens if and only if  $i_{F/K}(\sigma) \leq s$ , and this happens if and only if  $\sigma \in (G/H)_s$ .  $\square$

4.3.4. *Herbrand's theorem.* Now, we can prove our main result about Herbrand's function and the upper indexed filtration.

**Theorem 4.3.5.** *Assume that  $L/K$  is an almost monogeneous finite Galois extension of real-valued fields and  $F$  is the invariant field of a normal subgroup  $H \subseteq G = \text{Gal}(L/K)$ . Then,*

- (i)  $\varphi_{L/K} = \varphi_{F/K} \circ \varphi_{L/F}$ ,
- (ii)  $(G/H)^s = G^s H/H$  for any  $s \in [0, 1]$ .

*Proof.* (i) Let  $r \in [0, 1]$  and  $s = \varphi_{L/F}(r)$ . Then the degree of the composite at  $r$  equals to

$$\deg \varphi_{F/K}(s) \deg \varphi_{L/F}(r) = |(G/H)_s| \cdot |H_r|.$$

By Corollary 4.3.3, the latter equals to  $|G_r|$ , which is the degree of  $\varphi_{L/K}$  at  $r$ . Since both functions are piecewise monomial and are equal to 1 at 1, they coincide.

(ii) Choose  $r$  with  $s = \varphi_{F/K}(r)$ . Then  $(G/H)^s = (G/H)_r = G_r H/H$  by Corollary 4.3.3, where  $r = \varphi_{L/F}(t)$ . It remains to note that  $\varphi_{L/K}(t) = s$  by part (i) and hence  $G_t = G^s$ .  $\square$

**Remark 4.3.6.** In the classical situation, the second part of Theorem 4.3.5 is called Herbrand's theorem. However, the first part is, perhaps, even more important. In a sense, it shows that Herbrand's function is a reasonable invariant of an almost monogeneous extension  $L/K$ .

4.3.7. *Herbrand's function for non-normal extensions.* Assume that  $K$  is almost monogeneous. Then part (i) of Herbrand's theorem allows to extend the definition of Herbrand's function to non-normal separable extensions  $F/K$ . Namely, embed  $F$  into a finite Galois extension  $L/K$  and define  $\varphi_{F/K}$  to be the piecewise monomial function that satisfies  $\varphi_{L/K} = \varphi_{F/K} \circ \varphi_{L/F}$ . This construction is independent of the embedding  $F \hookrightarrow L$  since we can dominate two embeddings by a third one and then the compatibility follows from Theorem 4.3.5(i). In particular, in view of Example 4.2.9, Herbrand's function is defined for any finite separable extension  $L/\mathcal{H}(x)$ , where  $x$  is a point on a  $k$ -analytic curve.

#### 4.4. Other properties of $\varphi_{L/K}$ .

4.4.1. *Tame extensions.* Since the valuation of  $K$  is not discrete, if  $L/K$  is tame then any  $\sigma \in G$  satisfies  $i_{L/K}(\sigma) = 1$ . Indeed, it follows from the standard theory of tame extensions that either  $\sigma$  acts non-trivially on  $\tilde{L}/\tilde{K}$  or it acts on an element  $x \in L^\circ$  via  $\mu_n$  with  $n$  invertible in  $\tilde{K}$ . In the latter case we can multiply  $x$  by an element of  $K$  making  $|x|$  arbitrarily close to 1. It follows that  $i_{L/K}(\sigma) = 1$  in either case, and we obtain the following result.

**Lemma 4.4.2.** *If  $L/K$  is a tame Galois extension then  $G_s = 1$  for any  $s < 1$ . So, the ramification filtration is trivial, 1 is the only jump point, and the Herbrand's function is the identity.*

4.4.3. *The different.* In general, the different  $\delta_{L/K}$  of a finite separable extension of real-valued fields  $L/K$  can be defined as the zeroth Fitting ideal of the module of differentials  $\Omega_{L^\circ/K^\circ}$ . This requires some care since  $\Omega_{L^\circ/K^\circ}$  is only almost finitely generated, see [GR03, Section VI.6.3]. However, in the differential rank one case this simplifies and one can use the usual definition, namely

$$\delta_{L/K} = |\text{Ann}(\Omega_{L^\circ/K^\circ})|$$

whenever  $\Omega = \Omega_{L^\circ/K^\circ}$  is *almost cyclic*, i.e. for any  $r \in |L^\circ|$  there exists  $a_r \in \Omega$  such that  $\Omega/a_r\Omega$  is killed by any  $\pi \in L$  with  $|\pi| \leq r$ . Note that in this case  $\delta_{L/K}$  is the limit of annihilators of the elements  $a_r$ .

**Lemma 4.4.4.** *Assume that  $L/K$  is a finite separable extension of real-valued fields such that  $L^\circ$  is the filtered union of monogeneous subrings  $A_i = K^\circ[x_i]$ . Then  $\delta_{L/K} = \lim_i |f'_i(x_i)|$ , where  $f_i$  is the minimal polynomial of  $x_i$  over  $K$ .*

*Proof.* Kähler differentials are compatible with filtered colimits, hence  $\Omega_{L^\circ/K^\circ}$  is the filtered colimit of  $\Omega_{A_i/K^\circ} = K^\circ dx_i / K^\circ f'_i(x_i) dx_i$  and we obtain that  $\delta_{L/K} = \lim_i \text{Ann}(\Omega_{A_i/K^\circ})$  is as asserted.  $\square$

4.4.5. *Upper jumps and the different.* In the classical discrete-valued case, one can compute the different in terms of the ramification jumps, see [Ser79, Ch. IV, Prop. 4]. This extends to our case as follows.

**Theorem 4.4.6.** *Assume that  $L/K$  is an almost monogeneous henselian finite Galois extension of real-valued fields and the valuation of  $K$  is not discrete. Let  $r_0 > r_1 > \dots > r_n$  be the jumps of the ramification filtration,  $g_i = |G_{r_i}|$  and  $g_{n+1} = 1$ . Then  $\delta_{L/K} = \prod_{i=0}^n r_i^{g_i - g_{i+1}}$ , in particular,  $\delta_{L/K}$  is the coefficient of the linear part of  $\varphi_{L/K}$ , i.e.  $\varphi_{L/K}(t) = \delta_{L/K} t$  on the interval  $[0, r_n]$ .*

*Proof.* It suffices to establish the formula for  $\delta_{L/K}$  since the second claim then follows from the product formula in 4.1.9. Given  $s \in |L^\circ|$  choose an  $s$ -generator  $x_s \in L^\circ$  and let  $f_s(t)$  be its minimal polynomial over  $K$ . Then  $L^\circ$  is the filtered union of its subrings  $K^\circ[x_s]$ , and so

$$\delta_{L/K} = \lim_{s \rightarrow 1} |f'_s(x_s)| = \lim_{s \rightarrow 1} \prod_{\sigma \neq 1} |x_s - \sigma(x_s)|$$

by Lemma 4.4.4. By Lemma 4.2.5, the latter limit equals to  $\prod_{\sigma \neq 1} i_{L/K}(\sigma)$ . It remains to note that  $i_{L/K}(\sigma) = r_i$  if and only if  $\sigma \in G_{r_i} \setminus G_{r_{i+1}}$ .  $\square$

4.4.7. *Extensions of degree  $p$ .* For extensions of degree  $p$  there is a single break point  $r_0$ . Using the above theorem we see that  $\varphi_{L/K}$  is completely described by the different.

**Corollary 4.4.8.** *Let  $L/K$  be as in Theorem 4.4.6 and assume that  $[L : K] = p = \text{char}(\tilde{K})$ . Then  $\varphi_{L/K}$  has degrees 1 and  $p$  and the only break point is given by  $r_0^{p-1} = \delta_{L/K}$ .*

4.4.9. *Generalization to separable extensions.* If  $K$  is almost monogeneous then Lemma 4.4.2, Theorem 4.4.6 and Corollary 4.4.8 hold for any finite separable henselian extension  $L/K$ . This is deduced from the analogous results for the Galois closure  $F/K$  of  $L/K$  and the Galois extension  $F/L$ . More concretely, to extend 4.4.2 one uses that if  $L/K$  is tame then  $F/K$  is tame, and to extend 4.4.6 one uses that the different is multiplicative in towers of extensions.

4.5. **Relation to the profile function.** We conclude the paper with describing the profile function in terms of Herbrand's function at all non-rigid points. In addition, one can describe the limit behaviour of  $\varphi_f$  at wildly ramified points and type 2 points using logarithmic Herbrand's functions of valuation fields of height 2. Since we have not established the higher ramification theory in the latter case, we will provide the description and only outline the argument.

4.5.1. *Comparison theorem.* We start with the comparison at non-rigid points.

**Theorem 4.5.2.** *Assume that  $f: Y \rightarrow X$  is a generically étale morphism between nice compact curves. Then for any non-rigid point  $y \in Y$  with  $x = f(y)$ ,  $L = \mathcal{H}(y)$  and  $K = \mathcal{H}(x)$ , the profile function  $\varphi_y$  of  $f$  at  $y$  coincides with the Herbrand's function  $\varphi_{L/K}$ .*

*Proof.* Let  $Z \rightarrow X$  be the Galois closure of  $f$ . Since profile functions are compatible with compositions of morphisms and Herbrand's functions are compatible with compositions of field extensions, it suffices to prove the claim for ramified Galois coverings  $Z \rightarrow Y$  and  $Z \rightarrow X$ . Thus, we can assume that  $f$  is Galois. Shrinking  $Y$  and  $X$  we can then achieve that  $f$  splits into a composition of topologically tame morphisms and morphisms of degree  $p$ . In the first case, both functions are the identity by Lemmas 3.3.2 and 4.4.2. In the second case, the different  $\delta_{L/K}$  determines  $\varphi_{L/K}$  via Corollary 4.4.8, and it remains to prove that the same formula works for  $\varphi_y$ . If  $y$  is of type 2 then this is the separable case of Theorem 3.3.7. This implies that the same relation between  $\delta_f(y)$  and  $\varphi_y$  holds for any point of  $Y^{\text{hyp}}$  because both  $\delta_f$  and  $\varphi_f$  are piecewise monomial, see [CTT14, Corollary 4.1.8] and Theorem 3.4.8.  $\square$

4.5.3. *The radii of  $N_{f, \geq d}$ .* For the sake of completeness, let us explicitly express the radii of the sets  $N_{f, \geq d}$  in terms of the Herbrand's function.

**Theorem 4.5.4.** *Assume that  $f: Y \rightarrow X$  is a generically étale morphism between nice compact curves and  $\Gamma = (\Gamma_Y, \Gamma_X)$  is a radializing skeleton of  $f$ . Then*

(i) *Each  $N_{f, d}$  with  $d \notin p^{\mathbb{N}}$  is a union of edges and vertices of  $\Gamma_Y$ . So, its closure is a finite subgraph of  $\Gamma_Y$ .*

(ii) *The radius  $r_i$  of the radial set  $N_{f, \geq p^i}$  is computed as follows: if  $y \in \Gamma_Y^{(2)}$ ,  $L = \mathcal{H}(y)$  and  $K = \mathcal{H}(x)$ , then  $r_i(y)$  is the break point  $r$  of  $\varphi_{L/K}$  such that  $\deg \varphi_{L/K} < p^i$  precisely on the interval  $[0, r)$ .*

*Proof.* By Theorem 4.5.2,  $\varphi_{L/K}$  coincides with the profile function  $\varphi_y$ , hence its degree (or logarithmic derivative) equals to the profile of the multiplicity function by Lemma 2.3.6. The assertion of (ii) follows in an obvious way. Note also that the degrees of  $\varphi_{L/K}$  lie in  $p^{\mathbb{N}}$ , hence the multiplicity outside of  $\Gamma_Y$  takes values in  $p^{\mathbb{N}}$ . Using that the multiplicity is constant on the edges of  $\Gamma_Y$ , we obtain (i).  $\square$

4.5.5. *The limit behaviour at type 1 points.* Assume, now, that  $f$  is wildly ramified at  $y \in Y$  and  $x = f(y)$ . By [CTT14, Theorem 4.6.4],  $\delta_f$  has a zero at  $y$  whose order equals to the order of the log different  $\delta_{y/x}^{\log}$  of  $\mathcal{O}_y/\mathcal{O}_x$ . In particular, if  $I$  is an interval starting at  $y$  and  $I = [0, r_1]$  is a radius parametrization induced by a parameter  $t_y \in m_y \setminus m_y^2$  then  $\delta_f(r) = cr^{\delta_{y/x}^{\log}}$ . In [CTT14] one also gets rid of  $c$  by rescaling  $t_y$ , but we will need the following finer construction. Provide  $K_y = \text{Frac}(\mathcal{O}_y)$  with the valuation of height two composed of the discrete valuation of  $K_y$  with uniformizer  $t_y$  and the standard valuation on the residue field  $\mathcal{O}_y/m_y = k$ , and define  $K_x$  analogously. To any  $a \in |K_y^\circ|$  one can associate a monomial function  $\psi_y(a)$  on  $I$  as follows: find a representation  $a = ct_y^n$  with  $c \in k$  and set  $\psi_y(a) = |ct_y^n|$ . This construction extends to piecewise monomial functions as follows: if  $\varphi: |K_y^\circ| \rightarrow |K_x^\circ|$  is a piecewise monomial function with breaks at  $s_i$  then  $\psi_y(\varphi): I \times [0, 1] \rightarrow I \times [0, 1]$  is the piecewise monomial function with breaks at  $\psi_y(s_i)$  and the same degrees as  $\varphi$  on the intervals. It is easy to see that the correspondence  $\varphi \mapsto \psi_y(\varphi)$  preserves composition of functions locally at  $y$ .

Now we can describe the limit behaviour at  $y$ . Let  $\varphi_y$  be the *logarithmic* Herbrand's function of  $K_y/K_x$ . Then there exists  $r_0 \in (0, r_1]$  such that  $\psi_y(\varphi_y)$  coincides with  $\varphi_f$  on the subinterval  $(0, r_0)$  of  $I$ . As usual, the proof uses the splitting method: both functions are compatible with compositions of morphisms, hence passing to the Galois closure and using a  $p$ -Sylow subgroup in the decomposition group of  $y$  one reduces to the cases of tame morphisms and morphisms of degree  $p$ . The first case is, as usual, trivial. In the second case, both functions have a single break point, and proving that they are equal reduces to proving that  $\psi_y(\delta_{K_y/K_x}^{\log}) = \delta_f$ . This is done essentially by the same argument as used in the proof of [CTT14, Theorem 4.6.4], namely, both sides are expressed as  $|ht_y t_x^{-1}|$ , where  $h = \frac{dt_x}{dt_y}$ .

4.5.6. *The limit behaviour at type 2 points.* A branch of  $Y$  at a type 2 point  $y$  can be described by a type 5 point  $z$ , see [CTT14, Section 3.4]. The field  $\mathcal{H}(z)$  is a valued field of height two. If  $I = [y, y']$  is an interval in the direction of  $z$  then there is a natural map  $\psi_z$  that associates to elements of  $|\mathcal{H}(z)|$  monomial functions on  $I$ . Furthermore,  $\psi_z$  extends to piecewise monomial functions and  $\psi_z(\varphi_{\mathcal{H}(z)/\mathcal{H}(f(z))})$  coincides with  $\varphi_f$  on a small enough neighborhood of  $y$  in  $I$ . The arguments are the same as outlined in the previous section.

**Remark 4.5.7.** The results stated in the last two sections indicate that it is more natural to interpret  $\varphi_f$  as the logarithmic Herbrand's function. This is not essential when interpreting the single value of  $\varphi_f$  at a point  $y \in Y^{\text{hyp}}$ , but becomes visible in the study of asymptotic behaviour. The same observation holds true already for the different function, see [CTT14, Section 4.7.3 and Remark 4.7.4].

## REFERENCES

- [Ber93] Vladimir G. Berkovich, *Étale cohomology for non-Archimedean analytic spaces*, Inst. Hautes Études Sci. Publ. Math. (1993), no. 78, 5–161 (1994). MR 1259429 (95c:14017)

- [Ber04] ———, *Smooth  $p$ -adic analytic spaces are locally contractible. II*, Geometric aspects of Dwork theory. Vol. I, II, Walter de Gruyter GmbH & Co. KG, Berlin, 2004, pp. 293–370. MR 2023293 (2005h:14057)
- [CTT14] Adina Cohen, Michael Temkin, and Dmitri Trushin, *Morphisms of Berkovich curves and the different function*, ArXiv e-prints (2014), <http://arxiv.org/abs/1408.2949>.
- [GR03] Ofer Gabber and Lorenzo Ramero, *Almost ring theory*, Lecture Notes in Mathematics, vol. 1800, Springer-Verlag, Berlin, 2003. MR 2004652 (2004k:13027)
- [Ser79] Jean-Pierre Serre, *Local fields*, Graduate Texts in Mathematics, vol. 67, Springer-Verlag, New York-Berlin, 1979, Translated from the French by Marvin Jay Greenberg. MR 554237 (82e:12016)

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